Exact explicit vortex solution to the n-D averaged Burgers equations

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Abstract

Closed global analytical solutions to the statistically averaged n-dimensional (n-D) nonlinear Burgers equations and compressible 3-D vortex Helmholtz equations are obtained. These solutions provide a rigorous resolution to the fundamental, long-standing problem of statistical closure in turbulence theory. This is achieved by utilizing an exact explicit solution in Eulerian variables for the inviscid n-D Burgers equations in an unbounded domain, evaluated under a stochastic shift by a uniform n-D Wiener process.

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1. Introduction

The method of representing solutions to partial differential equations (PDEs) as the statistical expectation of functionals of random processes has a rich history dating back to the foundational work of Einstein, Feynman, Kolmogorov, and Kac. While this framework is well-developed for linear parabolic equations, it holds exceptional promise for the theory of nonlinear PDEs [1]–[10]. For instance, Constantine and Iyer (2008) [1] introduced a viscous regularization of the inviscid Burgers equation via stochastic Lagrangian paths.

The study of the inviscid Burgers equation itself has long been central to the analysis of mathematical turbulence and shock formation. In pioneering works, Sinai [11], [12] (see also, [13]) successfully utilized the Lax–Oleinik variational formulation to establish the statistical properties of shocks and to construct global solutions under random initial perturbations, represented explicitly in Eulerian variables for one space dimension (1-D).

In the present paper, we analyze the possibility of an effective viscous regularization for multidimensional vortex solutions in Eulerian variables. To do so, we develop an approach that fundamentally diverges from both stochastic Lagrangian models [1] and classical variational methods [11]–[13]. Our framework operates entirely within Eulerian coordinates and relies on the machinery of generalized functions (distributions) [14], [15]. We utilize a known [16] (see also [17]–[21]) exact analytical solution in Eulerian variables for the n-dimensional (n-D) inviscid Burgers equation in an unbounded domain, where the initial data enter explicitly into the argument of the Dirac delta function.

In an earlier work (1991) [16], the author obtained exact 2-D and 3-D vortex solutions, contrasting with the well-known vortex-free potential solutions of the n-D Burgers equations considered in the inviscid limit in [11]–[13]. By deploying an n-D uniform Wiener process directly within this Eulerian, distribution-based representation, we bypass the geometric complexities of multidimensional Lax–Oleinik minimizers. This allows us to derive explicit, closed-form analytical expressions for any single-point or multi-point statistical moments and spectra of the velocity and vorticity fields [16]–[21], thereby providing a rigorous mechanism for effective viscosity.

Indeed, by introducing new aspects of the Eulerian framework to the well-known stochastic methods [1]–[10], this work circumvents the cornerstone closure problem in turbulence theory, which has remained unresolved for more than a century [22]–[24]. Traditionally, this problem requires solving an infinite hierarchy of equations for turbulent moments. We show that combining an explicit exact solution of the n-D inviscid Burgers equation with a uniform Wiener process to model effective viscosity enables an exact solution to the following averaged n-D Burgers equation:

$$\frac{\partial \langle u_i \rangle}{\partial t} + \left\langle u_j \frac{\partial u_i}{\partial x_j} \right\rangle = \nu_E \frac{\partial^2 \langle u_i \rangle}{\partial x_j^2} \qquad (1)$$

The angle brackets in (1) denote statistical averaging over realizations of a random Gaussian, delta-time-correlated velocity field that simulates effective viscosity. Crucially, the divergence of both the velocity field itself and its average value in (1) is non-zero.

The classical closure problem arises from the quadratic nonlinearity of the hydrodynamic equations. Utilizing Eulerian variables to solve the inviscid n-D Burgers equations allows us to obtain an exact, explicit analytical solution for the averaged velocity field in (1) without the need to directly solve equation (1) or its corresponding infinite system of equations for higher-order moments.

To date, the application of effective viscosity or diffusion modeling to nonlinear equations [1]–[10] has been restricted to implicit schemes based on the Lagrangian representation of the underlying inviscid equations. For example, in, effective viscosity is introduced into the inviscid Burgers equation by shifting the deterministic trajectories of Lagrangian particles via a uniform Wiener process. That approach first derives an exact implicit solution of the deterministic Riemann–Hopf equation in Lagrangian variables, and subsequently applies a random trajectory shift to perform statistical averaging. The resulting solution for the Burgers equation retains an implicit dependence on the initial conditions due to the inherent nature of Lagrangian variables.

Conversely, our paper demonstrates that an explicit solution form in Eulerian variables equally permits the introduction of effective viscosity through a similar random shift of fluid particle trajectories and subsequent averaging over a uniform Wiener process. However, unlike the Lagrangian approach in, the resulting averaged solution satisfies the statistically averaged Burgers equation (1) rather than the standard deterministic one.

The remainder of this paper is organized as follows. In Section 2, we present the procedure for reducing the inviscid n-D Burgers equations to the averaged equation (1) within the Eulerian framework and

provide its explicit exact analytical solution. In Section 3, we derive exact solutions for the compressible inviscid vortex 3-D and 2-D Helmholtz equations. We then present a counterexample to the formal generalization of the Beale–Kato–Majda (BKM) criterion [25] to compressible hydrodynamics. While the classical BKM theorem guarantees that solutions to the incompressible Euler equations remain regular as long as the maximum supremum of the vorticity remains finite, we show that this does not extend straightforwardly to compressible media. Finally, we demonstrate that introducing any non-zero effective viscosity yields globally smooth vortex solutions for the corresponding averaged modification of the vortex 3-D Helmholtz equation.

2.Stochastic inviscid Burgers equation and the solution of the closure problem

Let us consider the following stochastic modification of the n-dimensional generalization of the Riemann–Hopf equations, which describes the inertial motion of fluid particles under a zero balance between external forces, friction, and pressure gradients [16]-[21]–

$$\frac{\partial u_i(\vec{x};t)}{\partial t}+\left(u_j(\vec{x};t)+V_j(t)\right)\frac{\partial u_i(\vec{x};t)}{\partial x_j}=0;n=1,2,3... \tag{2}$$

In equation (2), in contrast to the standard multidimensional Riemann–Hopf equations, we incorporate the additional advection of fluid particles within a random homogeneous velocity field $V_i\left(t\right)$. This velocity field models the dissipative effects induced by molecular (or turbulent, depending on the macro-scales of the compressible medium) viscosity. Under this framework, the following multidimensional relations are assumed to hold:

$$\begin{aligned}&\left\langle V_i\right\rangle=0;\\&\left\langle V_i(t_2)V_j(t_1)\right\rangle=2n\nu\delta_{ij}\delta(t_2-t_1);\\&w_i(t)=\int_0^t dt_1V_i(t_1);\left\langle \vec{w}^2\right\rangle=2\nu nt\end{aligned} \tag{3}$$

where $\vec{w}\left(t\right)$ represents a homogeneous n-D Wiener process. The global strong (classical) solution to equation (2), subject to condition (3), is obtained in the form [16]-[21]–:

$$\begin{aligned}&\left\langle u_i(\vec{x};t)\right\rangle=\frac{1}{(2\sqrt{\pi tn\nu_E})^n}\int_{-\infty}^{\infty}dw_1..\int_{-\infty}^{\infty}dw_nu_i(\vec{x}-\vec{w};t)\exp\left(-\frac{\vec{w}^2}{4\nu_Ent}\right);\\&\nu_E=0:u_i(\vec{x};t)=\int d^n\xi u_{0i}(\vec{\xi})\det\hat{A}(\vec{\xi};t)\delta\left(\vec{\xi}-\vec{x}+tu_0(\vec{\xi};t)\right);\\&\hat{A}(\vec{\xi};t)=\delta_{kl}+t\frac{\partial u_{0k}(\vec{\xi})}{\partial\xi_l}\end{aligned} \tag{4}$$

3.Exact solution to the 3-D compressible Helmholtz equations

Applying the curl operator to equation (2) yields the following modification of the 3-D compressible Helmholtz equation for the vortex (vorticity) field $\vec{\omega}$:

$$\frac{\partial \omega_i}{\partial t}+\left(u_j+V_j(t)\right)\frac{\partial \omega_i}{\partial x_j}=\omega_j\frac{\partial u_i}{\partial x_j}-\omega_i div\vec{u};$$
$$\omega_i=\varepsilon_{ijl}\frac{\partial u_l}{\partial x_j};i=1,2,3 \tag{5}$$

Equation (5) admits the following exact, globally smooth analytical solution for the statistically averaged three-dimensional vortex field:

$$\left\langle \omega_i(\vec{x};t)\right\rangle=\frac{1}{(2\sqrt{3\pi t\nu_E})^3}\int_{-\infty}^{\infty}dw_1.\int_{-\infty}^{\infty}dw_2\int_{-\infty}^{\infty}dw_3\omega_i(\vec{x}-\vec{w};t)\exp\left(-\frac{\vec{w}^2}{12\nu_E t}\right);$$
$$\nu_E=0:\omega_i(\vec{x};t)=\int d^3\xi\left(\omega_{0i}(\vec{\xi})+t\omega_{0j}(\vec{\xi})\frac{\partial u_{0i}(\vec{\xi})}{\partial \xi_j}\right)\delta\left(\vec{\xi}-\vec{x}+t\vec{u}_0(\vec{\xi};t)\right);i=1,2,3 \tag{6}$$

Similarly, to the velocity solution (4), expression (6) corresponds precisely to the compressible Helmholtz equation averaged over the Wiener fluctuations:

$$\frac{\partial\left\langle \omega_i\right\rangle}{\partial t}+\left\langle u_j\frac{\partial \omega_i}{\partial x_j}\right\rangle=\left\langle \omega_j\frac{\partial u_i}{\partial x_j}\right\rangle-\left\langle \omega_i div\vec{u}\right\rangle+\nu_E\frac{\partial^2\left\langle \omega_i\right\rangle}{\partial x_j^2} \tag{7}$$

Crucially, owing to the proposed viscosity modeling framework, the exact solution (6) to equation (7) is established bypassing the need to solve an infinite hierarchy of equations for higher-order moments. In the two-dimensional case, the averaged vortex field satisfies an exact solution to equation (5) where the first term on the right-hand side vanishes:

$$\left\langle \omega_i(\vec{x};t)\right\rangle=\frac{1}{(2\sqrt{2\pi t\nu_E})^2}\int_{-\infty}^{\infty}dw_1.\int_{-\infty}^{\infty}dw_2\omega_i(\vec{x}-\vec{B};t)\exp\left(-\frac{\vec{w}^2}{8\nu_E t}\right);$$
$$\nu_E=0:\omega_i(\vec{x};t)=\int d^2\xi\omega_{0i}(\vec{\xi})\delta\left(\vec{\xi}-\vec{x}+t\vec{u}_0(\vec{\xi};t)\right);i=1,2 \tag{8}$$

Note that, unlike the averaged solutions (4) and (6), the unaveraged exact solutions to equations (2) and (5)—defined via (4) and (6) without the stochastic Wiener shift—remain smooth only over a finite time interval $0\le t<t_{nC}$. This minimal critical time interval t_{nC} is determined by the roots of the following determinant equation:

$$\det \hat{A}(\vec{x}=\vec{x}_{(nC)};t=t_{nC})\equiv \det\left|\delta_{ij}+t\partial u_{0i}/\partial x_j\right|_{t=t_{nC};\vec{x}=\vec{x}_{(nC)}}=0;$$
$$n=1:\det\hat{A}=1+tdu_{01}/dx_1=0\Rightarrow t_{1C}=\min_{x_1=x_{1C}}\left(du_{01}/dx_1\right)^{-1};$$
$$n=2:\det\hat{A}=1+tdiv\vec{u}_0+t^2U_{012}=0;U_{012}\equiv\frac{\partial u_{01}}{\partial x_1}\frac{\partial u_{02}}{\partial x_2}-\frac{\partial u_{01}}{\partial x_2}\frac{\partial u_{02}}{\partial x_1};\quad(9)$$
$$n=3:\det\hat{A}=1+tdiv\vec{u}_0+t^2\left(U_{012}+U_{013}+U_{023}\right)+t^3\det\left(\partial u_{0i}/\partial x_j\right)=0$$

For instance, in the case of specific initial data $n=2:u_{0i}=\varepsilon_{ij}\partial\psi_0/\partial x_j;\varepsilon_{12}=-\varepsilon_{21}=1,\varepsilon_{11}=\varepsilon_{22}=0;\psi_0=a\sqrt{L_1L_2}e^{-x_1^2/L_1^2-x_2^2/L_2^2}$, equation (9) yields a finite minimal collapse time $t_{2C}=e\sqrt{L_1L_2}/2a$ at the elliptic curve $x_{1(2C)}^2/L_1^2+x_{2(2C)}^2/L_2^2=1$.

To derive the exact solutions to equations (2) and (5) in the explicit forms (4) and (6), we utilize the following fundamental identities based on the properties of the Dirac delta function [16]-[21]:

$$\frac{\partial\delta(\vec{y})}{\partial x_m}\equiv-A_{km}^{-1}(\vec{\xi};t)\frac{\partial\delta(\vec{y})}{\partial\xi_k};\vec{y}\equiv\vec{\xi}-\vec{x}+t\vec{u}_0(\vec{\xi});$$
$$A_{li}A_{lm}^{-1}=\delta_{lm}\quad(10)$$

$$\delta\left(\vec{\xi}_2-\vec{\xi}_1+t\left(\vec{u}_0(\vec{\xi}_2)-\vec{u}_0(\vec{\xi}_1)\right)\right)\equiv\frac{\delta\left(\vec{\xi}_2-\vec{\xi}_1\right)}{\det\hat{A}(\vec{\xi}_1;t)};\quad(11)$$
$$\hat{A}(\vec{\xi};t)\equiv A_{ij}=\delta_{ij}+t\partial u_{0i}(\vec{\xi})/\partial\xi_j$$

$$\frac{\partial\det\hat{A}}{\partial t}\equiv\frac{\partial u_{0m}}{\partial\xi_k}A_{km}^{-1}\det\hat{A}\quad(12)$$

$$\frac{\partial}{\partial\xi_k}\left(A_{km}^{-1}\det\hat{A}\right)\equiv0\quad(13)$$

The formal mathematical derivations of relations (10)–(13) are documented in detail in [17], [18]. Utilizing identities (10)– (13), the exact solution for the statistical expectation of the velocity field divergence is expressed as:

$$\frac{\partial\langle u_i(\vec{x};t)\rangle}{\partial x_i}=\frac{1}{(2\sqrt{\pi\nu_E nt})^n}\int d^n\xi\frac{\partial\det\hat{A}(\vec{\xi};t)}{\partial t}\exp\left(-\frac{\left(\vec{x}-\vec{\xi}-t\vec{u}_0(\vec{\xi})\right)^2}{4nt\nu_E}\right)\quad(14)$$

Let us evaluate the closed single-point moments of the field (14) in the limit $t\to t_{nC};\mathrm{Re}=L^2/\nu t_{nC}>>1$ [21]:

$$\Omega_n^m=\frac{1}{L^n}\int d^nx\left(\frac{\partial\langle u_i\rangle}{\partial x_i}\right)^m\cong\mathrm{Re}^{\frac{n(2m-3)}{6}};m=2,3,4..\quad(15)$$

In the statistical theory of fully developed turbulence at high Reynolds numbers, it is well established that the scaling exponents of intermittent fields exhibit non-linear behavior $\Omega_n^{2m} / \left(\Omega_n^m\right)^2 >> 1$[22], [23]. Indeed, from (15) we deduce:

$$\frac{\Omega_n^{2m}}{\left(\Omega_n^m\right)^2} \cong \mathrm{Re}^{n/2} >> 1, \mathrm{Re} >> 1 \tag{16}$$

Similarly, applying relations (10)–(13) yields the following representation for the enstrophy (the spatial integral of the squared vorticity field) corresponding to the exact solutions of the inviscid, compressible 3-D and 2-D Helmholtz equations at zero effective viscosity $\nu_E = 0$:

$$\begin{aligned}
&\tilde{\Omega}_3^2\left(\nu_E = 0\right) = \frac{1}{L^3}\int d^3x\vec{\omega}^2 = \frac{1}{L^3}\int d^3x \frac{\left(\omega_{0i}(\vec{x}) + t\omega_{0j}\partial u_{0i} / \partial x_j\right)^2}{\det \hat{A}\left(\vec{x};t\right)} \cong O\left(1/\left(t_{3C} - t\right)\right) \to \infty, t \to t_{3C};\\
&\tilde{\Omega}_2^{\ 2}\left(\nu_E = 0\right) = \frac{1}{L^2}\int d^2x\omega_3^2 = \frac{1}{L^2}\int d^2x \frac{\omega_{03}^2}{\det \hat{A}\left(\vec{x};t\right)} \cong O\left(1/\left(t_{2C} - t\right)\right) \to \infty, t \to t_{2C}
\end{aligned} \tag{17}$$

According to (17), the vortex field solutions (6) and (8) at zero effective viscosity preserve smoothness only within the bounded interval $0 \le t < t_{nC}, n = 2;3$. In the limit $t \to t_{2C}$, these exact solutions to the compressible inviscid Helmholtz equations develop a finite-time singularity. Paradoxically, the maximum modulus of the vortex field itself remains strictly bounded by a finite value for the 2-D case in solution (8) at $\nu_E = 0$:

$$\omega_3\left(\vec{x};t\right) \le \max_{\vec{x}}\left|\omega_{03}\left(\vec{x}\right)\right| \equiv M_2 = const \tag{18}$$

Similarly, for the three-dimensional case, the maximum modulus of the vortex field cannot attain an indefinitely large value at any finite evolution time, as the solution (6) at zero effective viscosity implies the upper bound:

$$\left|\vec{\omega}\left(\vec{x};t\right)\right| \le \max_{\vec{x}}\left|\vec{\omega}_0\left(\vec{x}\right) + t\left(\vec{\omega}_0\left(\vec{x}\right)\vec{\nabla}\right)\vec{u}_0\left(\vec{x}\right)\right| \equiv M_3 t < \infty, t < \infty \tag{19}$$

The constraints (18) and (19), however, fail to ensure the global regularity of solutions (8) and (6) at zero viscosity, as evidenced by the divergence of the enstrophy in (17).Conversely, for incompressible flows, Beale, Kato, and Majda (BKM) [25] proved a foundational theorem stating that the maximum norm of the vorticity controls the breakdown of smooth solutions to the 3-D Euler equations. Specifically, if a smooth solution loses regularity at a critical time, the maximum vorticity must grow without bound; equivalently, if the vorticity remains bounded, the smooth solution persists. Our explicit counterexample demonstrates that this classical paradigm does not extend straightforwardly to compressible media. This indicates a critical need to refine the formal generalizations of the BKM theorem when applied to the vortex flows of an ideal, inviscid compressible fluid.

Conclusion

In this work, we have constructed an analytical continuation of the explicit solutions to the multidimensional inviscid Burgers equation and the 3-D compressible Helmholtz equations in Eulerian coordinates. These solutions feature an explicit dependence on the initial conditions, and their statistical expectation over the uniform Wiener process yields a globally smooth velocity field entirely free of shock-wave discontinuities.

Rather than merely offering a refinement or a minor alternative to the Lagrangian framework developed by Constantin and Iyer, this work establishes a novel direction within the classical theory of stochastic PDEs. Our approach can be formally interpreted as the Eulerian dual (mirror image) to the Lagrangian path method. The resulting closed description of the evolved mean velocity and vorticity fields does not coincide with the profiles generated by the standard deterministic viscous Burgers or Helmholtz equations. Instead, it provides exact solutions to a novel class of averaged equations that naturally incorporate macroscopic turbulent transport.

Consequently, we circumvent the century-old closure problem of turbulence theory without relying on an infinite, open-ended hierarchy of moment equations. While individual deterministic realizations of the inviscid flow develop gradient catastrophes and shocks at distinct times, the introduction of the Eulerian stochastic Wiener shift acts as a powerful regularizing mechanism upon averaging. Furthermore, this framework challenges the long-standing paradigm that explicit tracking of initial data in non-linear hydrodynamics is exclusive to Lagrangian trajectories, opening new analytical avenues for the global study of compressible, high-Reynolds-number turbulent flows.

Additionally, a prevailing paradigm in non-linear hydrodynamics holds that representing the explicit time evolution of the inviscid Burgers equation in Eulerian coordinates with clear initial-data dependence requires the Lax–Oleinik variational principle—the Eulerian analogue to the

method of characteristics. The practical application of that variational framework, however, inherently necessitates numerical methods and fails to yield explicit closed-form analytical expressions for the single-point and multi-point statistical moments of the velocity and vorticity fields. By contrast, the distribution-based framework developed in [14–21] and presented in this paper successfully achieves these explicit moment formulations by leveraging the mathematical machinery of generalized functions.

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